What is the Riemann Hypothesis?

Fintan Hegarty

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Student ID: 04436768
Supervisor: Goetz Pfeiffer

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of BSc (Honours) in Financial Mathematics and Economics is entirely my own work and has notbeen taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed:

Fintan Hegarty

Id No: 04436768

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Abstract

A friend of mine gave me a copy of duSautoy's *Music of the Primes* to read over the summer. It's a beautiful book detailing the history behind the notorious Riemann's Hypothesis. Unfortunately, the book dwelt more on the fascinating history than the specifics of the mathematics of the problem. I reasoned that the final year project would provide the perfect opportunity to permit further investigation, without feeling guilty for allowing the pursuit of what would otherwise have been, essentially, a dilletantish endeavour.

Since I had a sneaking suspicion that I would not make any advancements towards a proof of the Hypothesis, I decided instead that I would attempt to provide an explanation of the Hypothesis, aimed at a 'jury of my peers'. I am much indebted to my classmates for their views on what the average maths student ought to have been taught, and what they would probably have forgotten! At the end I have also attempted to touch upon some of the more intricate mathematics that were at the heart of Riemann's paper from which his Hypothesis came.

According to Andrew Weil himself *'The greatest problem for mathematicians now is probably the Riemann Hypothesis. But it's not a problem that can be simply stated.'*

So here is my attempt at an explanation.

Contents

1 The Primes

'God may not play dice with the universe, but something strange is going on with the prime numbers.' - Paul Erdos

2,3,5,7,11,13,17,19,23...
We know these are the primes - they are the very building blocks of arithmetic, as the fundamental theorem of arithmetic states that every integer can be expressed as a unique product of primes.
There exists an infinite number of them - this was proven by Euclid thousands of years ago.
His proof ran simply thus:
Suppose N is the maximum prime. Then if you take the product of all the primes up to and including N and 1 to it - you have a number which is not divisible by any of those primes. Thus, by the Fundamental Theorem of Arithmetic, this is a new (larger) prime.
Interestingly we can also construct a sequence of numbers of arbitrary length without a prime.
If we want a sequence of at least N consecutive non-prime numbers we simply calculate $N!$, and while $N!+1$ is prime, the $N!+2$ will is divisible by 2, $N!+3$ is divisible by 3,...$N!+100$ is divisible by 100.

But for all that is known about the primes and despite the thousands of years of research - there was still no formula for accurately predicting the n^{th} prime. We have, of course, the familiar sieves such as the Sieve of Eratosthenes, where one goes through the laborious process of writing out a list of numbers and then removing any numbers with factors, beginning by crossing out any number that divides 2, then any nunber that divides 3, then 5, etc... leaving us, of course, with only primes. But this is more to do with brute calculation, and necessitates little of the adroitness with which to satisfy the mathematical aesthete.

There is an interesting tale in psychologist Oliver Sacks' renowned book of curious casefiles; of two autistic twins who played a game where they called out prime numbers to each other, and if one made a mistake the other would break it into its prime factors. Sacks tells of how, when he first met them, they were playing with 12-digit primes, but after he joined in (using a book of primes), they advanced slowly, until when he left them they had reached twenty digit primes - well beyond the scope of Sacks' tables. Sadly, they were separated in an attempt to 'integrate' them into society. No one has any idea how they were coming up with these numbers, whether they just 'saw' the numbers, or whether they were using some sort of strange modular arithmetic as it was theorised they did for working out what day of the week any date within a few centuries fell on. The rest of their interesting story, and many others can be found in Sacks' *The Man Who Mistook his Wife for a Hat.*

But for a more mathematical approach we must turn elsewhere, to a young Karl Friedrich Gauss who was given a table of logarithms one year as a gift. At the back, this book contained a table of primes, as a curiosity.
But Gauss, being Gauss, found a connection between the primes and the natural logarithms.[1]
Instead of trying to predict the next prime though, he instead ventured a formula for calculating their distribution. The notation $\pi(x)$ is used to express the number of primes up to a given 'x'.
Gauss conjectured that

$$\pi(N) \sim \frac{N}{\ln N}$$

Obvious consequences of this Prime Number Conjecture would be that: The probability that N is prime $\sim \frac{1}{\log N}$
And the N^{th} prime number $\sim \frac{N}{\log N}$.
Following closely on the heels of this conjecture came another similar estimate.
It is not entirely unexpected that after Gauss came up with his conjecture that this one would follow:

$$\pi(x) \sim \int_{t=0}^{x} (1/\log t)\, dt$$

And as it happens, this approximation is significantly more accurate. However for clarity's sake, this integral is often shorted simply to $Li(x)$. Gauss observed that $Li(x)$ seemed to be significantly more accurate than $N/\log N$, and that the former seemed to consistently overestimate, while the latter underestimated.
And there we shall leave the primes for the moment.

[1] All logs referred to in this paper are natural logs.

2 The Zeta Function

'Mathematicians have tried in vain to this day to discover some order in the sequence of prime numbers, and we have reason to believe that it is a mystery into which the human mind will never penetrate.' - Euler

Here is the Harmonic Series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}++\frac{1}{5}+\frac{1}{6}+\frac{1}{7}...$$

Nicole Oresme proved that this series diverges to infinity.

$$1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+...$$

is greater than the series summing

$$1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+...$$

and since the latter tends to infinity, it is clear that the harmonic series does too, albeit at a decelerating rate.
The harmonic series has interesting properties relating to the production of musical notes, but unfortunately they are of little relevance at the moment and we must refrain from digressing.

From the Harmonic series arose the Basel problem:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...$$

There was little difficulty in seeing that the series was converging between 1.644 and 1.645 - but where exactly? Mathematicians were not simply satisfied with an irrational approximation - they wanted to find a closed form for the series. But it was a long time in coming. Even the Bernoullis tried and failed.
But it was eventually solved by Leonard Euler, who plays no small part in this tale.

Here is an outline version of his proof:

We begin with an n^{th} degree polynomial $p(x)$ such that $p(x)$ has non-zero roots $a_1, a_2..., a_n$ and $p(0) = 1$.
So $p(x)$ can be written in the form

$$\left(1 - \frac{x}{a_1}\right) \times \left(1 - \frac{x}{a_2}\right) ... \times \left(1 - \frac{x}{a_n}\right)$$

Then Euler made the claim that 'what holds for a finite polynomial holds for an infinite polynomial.'
Then he took the sin function, as derived from the Taylor series:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^{k+1}\frac{x^{2k+1}}{(2k+1)!} + ...$$

Then

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$$

The roots of $\sin(x)$ are at $\pm k\pi$ for $k = 1, 2, ...$ since these send $\frac{\sin(x)}{x}$ to 0.
So writing this $\frac{\sin(x)}{x}$ as a product of factors is

$$\left(1 - \frac{x}{\pi}\right) \times \left(1 + \frac{x}{\pi}\right) \times \left(1 - \frac{x}{2\pi}\right) \times \left(1 + \frac{x}{2\pi}\right) \times \left(1 - \frac{x}{3\pi}\right) \times \left(1 + \frac{x}{3\pi}\right) \times ...$$

$$= \left(1 - \tfrac{x^2}{\pi^2}\right) \times \left(1 + \tfrac{x^2}{\pi^2}\right) \times \left(1 - \tfrac{x^2}{4\pi^2}\right) \times \left(1 + \tfrac{x^2}{4\pi^2}\right) \times \left(1 - \tfrac{x^2}{9\pi^2}\right) \times ...$$

This will give us an infinite number of terms but we can see that they go something like this:

$$1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + ...\right) x^2 + ...$$

So if we equate this with the

$$\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ...$$

expression, and we match the two x^2 coefficients we get

$$\frac{-1}{3!} = -\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + ...\right)$$

$$\left(-1/\pi^2\right)\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...\right)=\frac{-1}{3!}$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...=\frac{\pi^2}{6}$$

The Harmonic Series and the Basel Problem are both examples of what is called the zeta function [2]$\zeta\left(s\right)$.

$$\zeta\left(s\right)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}...$$

The slightly oversimplified version shown of Euler's solution to the Basel problem, shows the basic idea which Euler used to find closed form solution for $\zeta\left(2n\right)$ for every integer n up to 26. His formula

$$\zeta\left(2k\right)=\left(-1\right)^{k-1}\left(2^{2k-1}\right)\left(B_{2k}\right)\pi^{2k}/\left(2k!\right)$$

works for every even value for ζ but no formula has been found which will give a closed for $\zeta\left(s\right)$ when s is odd. We can calculate them to as many decimal places as we wish, but no simple closed form has been found.
Now... what has this $\zeta\left(s\right)$ got to do with the primes?
A couple of things - here is the first connection, as shown by Euler.

$$\zeta\left(s\right)=1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\frac{1}{9^s}+\frac{1}{10^s}+\frac{1}{11^s}+\frac{1}{12^s}+\frac{1}{13^s}+...$$

$$\frac{1}{2^s}\zeta\left(s\right)=\frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{10^s}+\frac{1}{12^s}+...$$

$$\left(1-\frac{1}{2^s}\right)\zeta\left(s\right)=1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{9^s}+\frac{1}{11^s}+\frac{1}{13^s}+...$$

[2]The convention is to use s instead of n following the example of the protagonist of this paper.

$$\left(\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta\left(s\right)=\frac{1}{3^s}+\frac{1}{9^s}+...$$

$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta\left(s\right)=1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+...$$

Similarly

$$\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta\left(s\right)=1+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}+...$$

And we can see that what we end up with is

$$...\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta\left(s\right)=1$$

So, where p are the primes

$$\zeta\left(s\right)=\prod\left(1-\frac{1}{p^s}\right)^{-1}$$

And finally we have Euler's Product Formula

$$\zeta\left(s\right)=\sum_{n=0}^{\infty}n^{-s}=\prod_{p=0}^{\infty}\left(1-\frac{1}{p^s}\right)^{-1}$$

This is actually another proof that there is an infinite number of primes - if there were not, then the $\prod_{p=0}^{\infty}\left(1-\frac{1}{p^s}\right)^{-1}$ could not equal the infinite sum.

So there we have our fist connection between the zeta-function and the primes.
And we have also introduced our function - but what of the argument s? What values can it have?
Well, clearly - if $s=1$, our series diverges. When s is greater than 1 the series converges.

Here is a graph of the zeta function for arguments greater than 1:

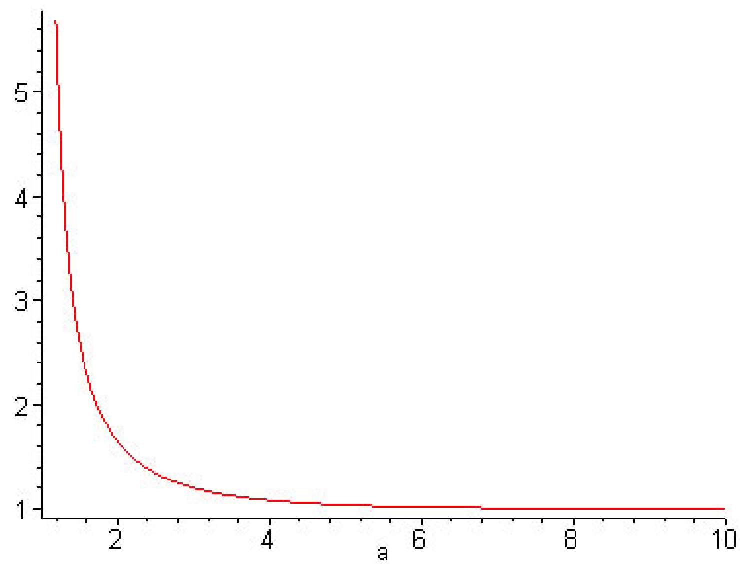

Figure 1: $\zeta\left(s\right) fors > 1$.

What though, if x is 0, or less than 0? Well,

$$\zeta\left(0\right) = 1 + \frac{1}{2^0} + \frac{1}{3^0} + \frac{1}{4^0} + \frac{1}{5^0} + \frac{1}{6^0} + \frac{1}{7^0} + ...$$

And since $x^0 = 1$, this sum diverges also.
If s is negative, then it's even worse: for example if we take $s =$ -2, $\frac{1}{x^{-2}} = x^2$ so you're adding a series of numbers which will diverge much faster than for $x = 1$.
What if s lay between 0 and 1? If $s = \frac{1}{2}$ say? We would be adding the sum of the reciprocals of all the square roots. Well, we know the harmonic series diverges, and if we take any integer x in the harmonic series, the corresponding inverse of $\sqrt{x}$ would be greater than the inverse of x itself.
For example if $x = 4$, $\frac{1}{4} < \frac{1}{\sqrt{4}} = \frac{1}{2}$.
So $\zeta\left(\frac{1}{2}\right)$ diverges faster than $\zeta\left(1\right)$ also.
So can we say that the domain of the zeta-function is the integers greater than 1?
No!

Let's imagine...

3 A Power Complex...

'i^2 - *Keepin' it Real!* '- *Anon.*

But what if our s is an irrational, or a complex number?
To recall briefly - suppose we take the irrational $x^{\sqrt{2}}$ first. We could use the sequence $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}$ which converges to $\sqrt{2}$ and correspondingly raising x to the sequence of powers will converge to the actual value of $x^{\sqrt{2}}$.
For complex numbers we must use the exponential function. [3] First we will raise e to a complex power with real part zero and then show how that can be expanded for any other number.
Here is the definition of e^z:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + ...$$

This infinite sum *eventually* converges for every number.
For example - let $z = \pi i$ Rounding off slightly this gives us:

$$e^{\pi i} = 1 + \frac{3.14159i}{1} - \frac{9.8696}{2} - \frac{31.00628i}{6} + \frac{97.4091}{24} + \frac{306.0197i}{120} - \frac{961.38432i}{720} - ...$$

The real part converges to -1 and the imaginary part converges to 0.
This is, of course, the familiar $e^{\pi i} = -1$.
Now, for other complex powers...
$a = e^{\log a}$ so $a^x = e^{x \log a}$
So letting z and w be our complex numbers - then $z^w = e^{w \log z}$

So, to calculate, say;
$(-4 + 7i)^{2-3i}$
First calculate the log of $-4 + 7i = 2.0872 + 2.0899i$
Multiply by w $(2.0872 + 2.0899i) \times 2 - 3i = 10.444 - 2.0817i$
Then raise $e^{10.444-2.0817i} = -16793.46 - 29959.4i$

Papers on problems such as these, by Cauchy amongst others, were what our protagonist was studying in the year before he travelled to Gottingen to do his doctoral thesis under none other than Gauss himself.

[3] We can also use this method for raising to irrational powers, but the converging sequence method used above is interesting enough to merit a mention.

4 Riemann the Navigator

'If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?' - David Hilbert

Recall our ζ function.
It was reasoned that the function $\zeta(s)$ could have values only when $s > 1$.
Riemann, when introduced to this new function, was still thinking about those somewhat revolutionary imaginary numbers, and would perhaps have fed those new arguments into the zeta-function, almost without thinking, just to see what they would produce.
Now, unfortunately, mapping a diagram of complex arguments to complex values would require a graph in four dimensions. This poses a problem.
But just as the two-dimensional shadow of a three-dimensional object can give us an idea, albeit an incomplete one, of the object's shape, we can use a similar notion to study an incomplete shadow of these four-dimensional graphs.
Riemann came up with this graph of the zeta-function:

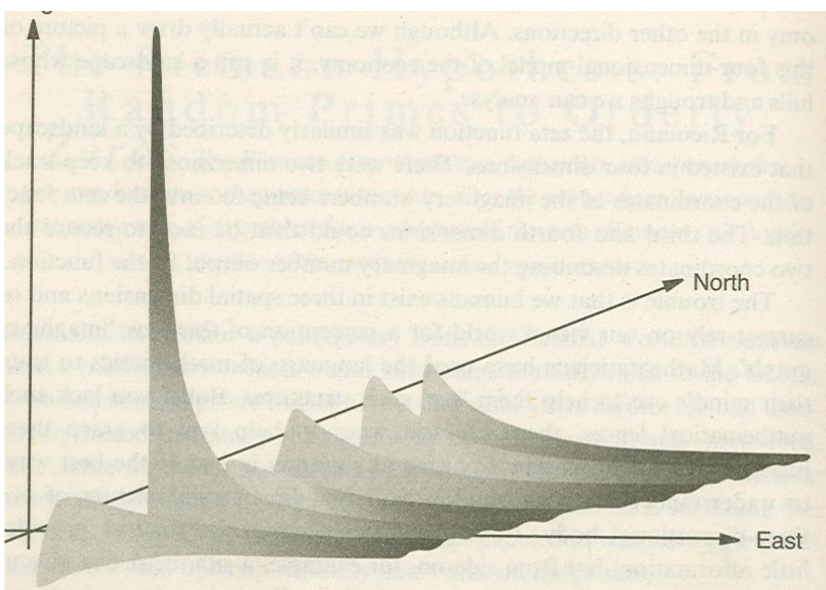

Figure 2: Riemann's ζ landscape

Now, the height above each imaginary number on the map shows the result

of feeding that number into the function. As with any shadow, we lose some details, but there is information to be garnered nonetheless.
If we imagine ourselves as explorers in this landscape we notice a few things. If we look to the east we can see that the landscape levels off to a flat plane at height 1 above sea level. But if we face west we see a series of hills running from north to south, with the image reflected in itself about one infinitely high peak on the x-axis at 1.
But what if we continue on past these peaks? The points to the east of the real number 1 are undefined by the zeta function, but Riemann saw no reason why the landscape should stop. What was there to prevent the map continuing into the interval from zero to 1 and on through the negative numbers? And so Riemann devised a way to calculate the values of the zeta function where $s < 1$.
For the interval 0 to 1:
Consider the function

$$\eta\left(s\right) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s ...}$$

We can see that this is more likely to converge than $\zeta\left(s\right)$.
But $\eta\left(s\right)$ can be written as

$$\left(1 - \frac{2}{2^s}\right)\zeta\left(s\right)$$

or

$$\zeta\left(s\right) = \eta\left(s\right)\left(1 - \frac{1}{2^{s-1}}\right)$$

This means that using the values for $\eta\left(s\right)$ it's possible to calculate values for the zeta-function in the domain between zero and one, even though it shouldn't, by all rights, converge there.
And what of $s \leq 0$?
This formula which gave $\zeta\left(1-s\right)$ in terms of $\zeta\left(s\right)$, was one of the main results of his 1859 paper

$$\zeta\left(1-s\right) = 2^{1-s}\pi^{-s}\sin\left(\frac{1-s}{2}\pi\right)\left(s-1\right)!\zeta\left(s\right)$$

And so, Riemann was able to graph the zeta-function over the entire complex plane, with a single pole at $s = 1$.

Here is a graph of the argument plane, centred on the origin. The points where the four colours meet are called the 'zeros' of the function. Filling any of these zeros into the ζ function will return a value of zero.

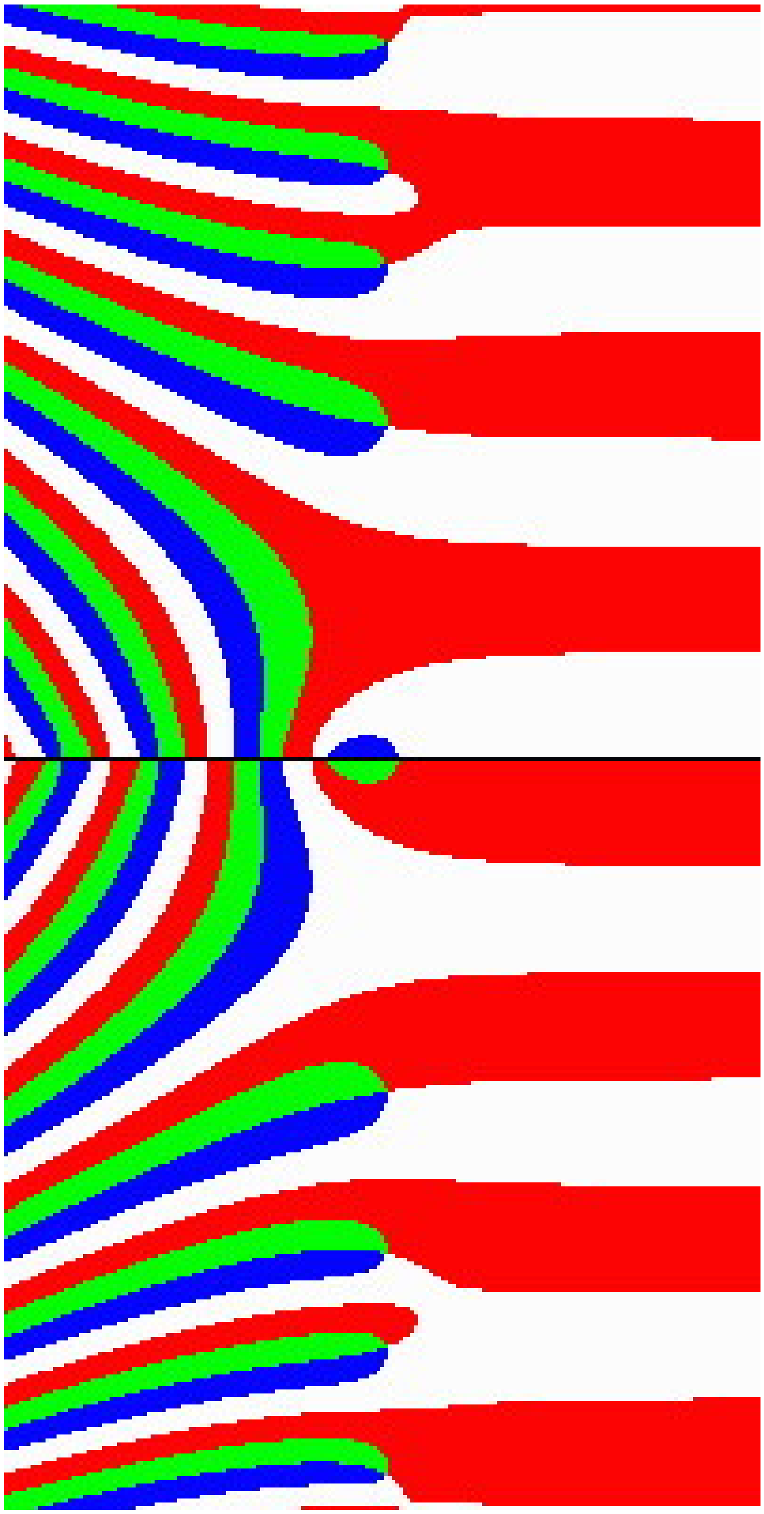

Figure 3: $\zeta(s)$ the argument plane.

Riemann found that just as the roots and poles of a polynomial can be used to graph the function according to the fundamental theorem of algebra, so too can its zeros and single pole be used to find a unique graph of the zeta-function.
And since both these zeros and the primes generated this same landscape, Riemann reasoned there must be some connection between them.
In fact, whereas Gauss had only been able to estimate $\pi(n)$, Riemann was now able to come up with an exact formula to calculate the number of primes. His formula consisted of two parts. The first was $R(n)$ which was a refinement of Gauss' guess, but was still producing errors. This foray of Riemann's into the imaginary landscape, had, in fact, provided him with a way to correct these errors!

Euler had discovered that if you fed an imaginary number into the exponential function the outcome was a sine wave. So the usual rapid climb of the function would be transformed into an undulating graph, similar to a sound wave.
Riemann saw how each of the points on the landscape could be transformed using the zeta-function to produce its own little wave, like a variation on the sine function. And for a given zero, the higher it was located above the x-axis, the faster its corresponding wave would oscillate.
If we use Berry and duSautoy's analogy of imagining the waves as sound waves, this means that the higer the point, the higher the note the wave would produce.
But how is this helpful for counting the primes?
Riemann discovered that by adding to his $R(n)$ function the heights of each wave above n, he could remove the error, and calculate the exact number of primes.

Riemann explored the zeta landscape a little further, to get a feel for how the zeros are laid out, and to see which zeros would have a greater effect in his formula. The trivial zeros, the ones located at -2,-4,-6... etc, could be ignored, because they were not analytic when fed into the expansion function. Or, to use the nicer musical analogy - since they were on the x-axis, they had a volume of zero, and thus - no effect!
The other zeros however all seemed to be lining themselves up along a vertical line through real part $= 1/2$. They would all have the same 'frequency'. Here is a another graph of the *arguments* of the zeta function, but this time

we are only dealing with the arguments which return either a real or an imaginary number. For clarity's sake we have omitted all arguments which return complex numbers:

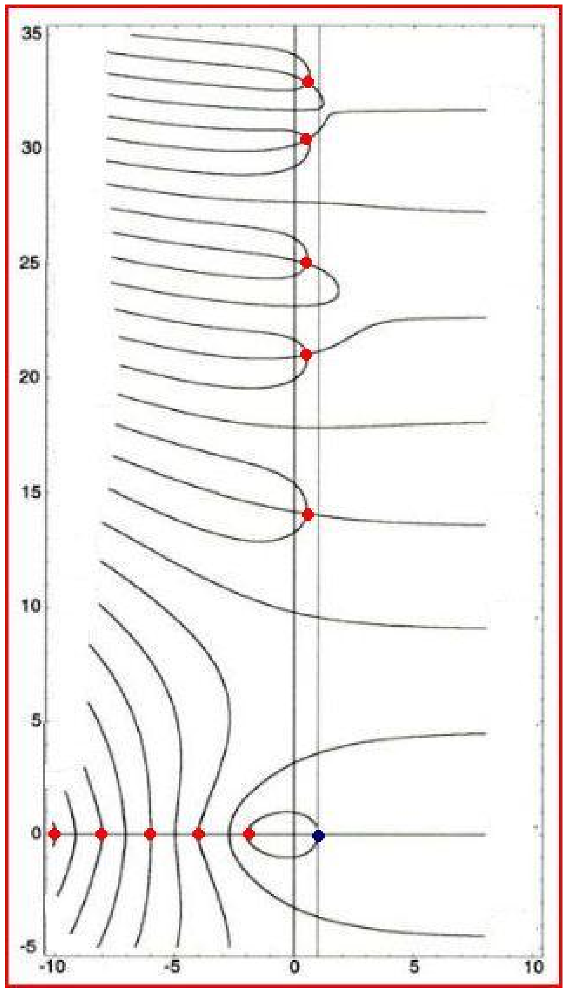

Figure 4: $\zeta(s)$ the argument plane.

Imagine you are standing on the point -2. Since this is a trivial zero, the function returns a value of zero and so we shall mark this spot with a little dot. (Note: if you fill -2 into Riemann's functional equation you end up with $\sin 2\pi$ on the right-hand side of the equation, which renders it equal to 0)

If you proceed due west along the x-axis from this point, ie. decreasing the real part of the argument, the returned function value will increase slowly until just after you pass 2.7, and if, from there on you continue due west, the values will decrease slowly again until you reach another trivial zero at x = 4, which again, is not analytic in the functional equation. And this pattern will just repeat itself as the arguments move from trivial zero to trivial zero. However, if instead of continuing west, you instead traverse the curve to the north, the function will go on increasing, ever so slowly, and the function value returned will tend slowly towards 1, but only after you've been travelling for an infinite length of time as

$$\zeta(\infty) = 1 + \frac{1}{2^\infty} + \frac{1}{3^\infty} + \frac{1}{4^\infty} + \frac{1}{5^\infty} + \frac{1}{6^\infty} + \frac{1}{7^\infty} + ... = 1$$

And supposing you 'reach infinity', and you then turn and come back along the real line from the east. (Think of infinity as a point on the Riemann sphere - you can return from any direction.) As we saw in the earlier graph of the real plane the function value increases slowly and then starts to increase very rapidly as you approach 1. To try and give a little perspective recall that it will return a value of 1.644 as we pass the argument 2 (recall our Basel Problem), and then as s approaches 1 the function value flies off to infinity.

If from 1 you continue straight along the x-axis, the graph of the function values inverts and you have massive negative values which decrease rapidly, returning a value of -0.5 when the argument is 0, because filling in 0 into

$$\zeta(s) = \eta(s)\left(1 - \frac{1}{2^{s-1}}\right) = \zeta(0) = \eta(0)\left(1 - \frac{1}{2^{0-1}}\right) = -0.5$$

And back to a value of 0 when we get back to -2.
Or if you take either of the tracks *around* the x-axis, then the function values

will travel up or down the *imaginary* axis, depending which track you take, and both of course, reach 0 as the tracks return to -2.

If we consider the zero at -4 you can continue with arguments going west along the x-axis, which will return values on the real line. Or you could take the upward path, which would return increasing imaginary values. Notice that, if we consider the function values - the zeros are where the x-axis and y-axis meet. So, from any zero, there will be more than one direction in which you can travel.

There is no zero at argument -5 because letting $s = -5 + ti$ will return a real value, and does not cause an intersection of the two axes in the value plane.

And - we can see the other zeros, the non-trivial ones, the ones that are not negative even integers, all *appear* to lie along the line $\frac{1}{2}$ + ti
This is *Riemann's Hypothesis*:

ALL NON-TRIVIAL ZEROS OF THE ZETA FUNCTION HAVE REAL PART ONE-HALF

There you have it.

The line $\frac{1}{2}$ + ti is called the *critical line*. Do all $\frac{1}{2}$ + ti return 0? No, clearly not. Here is a graph of the *values* of the zeta function with arguments $\frac{1}{2}$ + ti - it is not a graph of the arguments as we had in the previous diagram:

Here we can see that the argument $\frac{1}{2}$+ti = 0 on more than one occasion, but definitely not for all values of t .

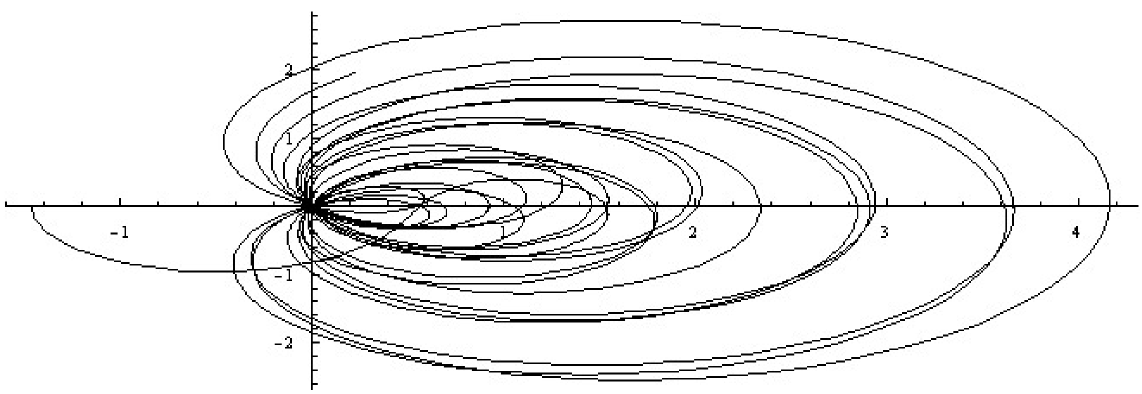

Figure 5: $\zeta\left(s\right)$ where $s = \frac{1}{2} + t$i

5 The Aftermath

'Statistics means never having to say you're certain.' - Anon.

Now this paper shall attempt to illustrate very briefly, what progress has been made into proving the Hypothesis, and how infuriatingly close mathematicians seem to have come, without actually having been able to complete the proof.

Riemann's paper only mentioned the Hypothesis briefly. He acknowledged that he had abandoned his search for a proof after 'a few fleeting vain attempts', as it was not necessary for the main aim of his paper. He assumed it was true and used it to give a conditional proof of Gauss' Prime Number Conjecture.
Nor did he give any indication of how he had managed to calculate the handful of zeros he did have, leaving some of his compatriots amazed at his seemingly visionary result, and leaving others a little more sceptical. This lack of transparency was perhaps reminiscent of Gauss, who would remove the 'scaffolding' that he had used, and only present the final result in his paper. Gauss had, for example, used the notion of imaginary numbers himself, to formulate one of his papers, but he omitted any mention thereof in the final draft and presentation for fear of causing undue controversy amongst his more conservative contemporaries.

Riemann died of tuberculosis aged only 39. His housekeeper, faced with the untidy mess of Riemann's work that he had left behind, burned some of his papers before Riemann's colleagues stopped her. Riemann's wife also took many of his papers for some sentimental value. And while most of these were eventually handed back to the mathematicians, some papers were inevitably lost - the most noteworthy of these being a little black notebook which Riemann had been known to carry around with him in the months prior to his 1859 paper.
And while Riemann's claims that he had a proof of the Hypothesis but was not quite yet ready to publish might put one in mind of Fermat's 'marginal' proof - it was known that Riemann was a perfectionist and would not publish anything which could not be wholly defended. And, as Siegel discovered many years later, when going through Riemann's papers containing the cal-

culations for those hidden results, Riemann had, in fact, discovered a lot more than had come to light. So it is not entirely impossible that Riemann did indeed have a proof, but that it has been lost...

In 1885 a Dutchman named Stieltjes claimed to have found a proof, but despite much hounding from his peers, failed to deliver. A prize was then offered for a proof, or something close to it. It was claimed, not by Stieltjes, but by Hadamard, whose work fell short of proving the Hypothesis, but he did manage to prove Gauss' conjecture, without making Riemann's assumption that the Hypothesis would hold.

And now with the Prime Number Theorem proven, the limelight of number theory fell upon Riemann's Hypothesis. This was due largely to Hilbert's famous lecture at the turn of the century wherein he challenged the mathematical world with 23 problems. And while most of the problems were quite open-ended and had more of a philosophical nature - Riemann's Hypothesis, the eighth problem, was quite definite.

So, with so many mathematicians now focused on it, it is unsurprising that some progress was made. Landau and Bohr proved that most of the zeros lie in an infinitely small strip about the critical line, Hardy proved that infinitely many of the zeros lie on the line - which was, admittedly, a substantial improvement on the 71 zeros which had been previously calculated, but still not enough to prove the Hypothesis.

Littlewood's supervisor gave him the Hypothesis to prove as a summer project. And while this was probably a little ambitious, it was not entirely without result. He discovered that there was a relationship between the primes and the zeta function! However, as we know, this had already been proven. Hardy recognised Littlewood's potential though, and thus began an extremely productive collaboration. Littlewood proved that two of Gauss' conjectures relating to the Prime Number Theorem were false; the more notable of these being that $Li\,(x)$ did NOT, in fact, always overestimate the number of primes. This was significant because when Littlewood began his work it was known that Gauss' conjecture held for up to ten million. But Littlewood's result showed the mathematical world that an overwhelming body of evidence can not be enough to guarantee truth. In fact, it was calculated that the conjecture would first fail at $e^{e^{e^{79}}}$, which is known as Skewe's number, and it is approximately equal to $10^{10^{8.85\times10^{33}}}$. (As of 2007, this bound has actually been reduced to a mere 1.397×10^{316}.)

But the warning of Littlewood's result bids mathematicians to be very wary of making assertions based on statistical evidence alone...

One day Hardy received a letter from an Indian clerk named Ramanjuan, which, amidst a jumble of other equations and a claim of a formula for calculating the primes there lay the equation:

$$1 + 2 + 3 + 4 + ... = -1/12$$

It is not surprising that most mathematicians empathised with the words of one Mr. Hill 'Mr. Ramanjuan has clearly fallen into the pitfalls of the very difficult subject of the divergent series.'
But Hardy and Littlewood realised that if you rewrote the equation as:

$$1 + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \frac{1}{4^{-1}} + \frac{1}{5^{-1}} ...$$

Ramanjaun had calculated $\zeta(-1)$. Hardy and Littlewood arranged for him to be transported to Cambridge where they were working. It turned out that Ramanjuan had, in fact, discovered all by himself practically everything that has been mentioned previously in this paper and much much more, but unfortunately this was some 150 years after the discoveries had already been made. He claimed that the goddess Namagiri showed them to him in his dreams. Unfortunately, she did not seem to be so keen on providing him with proofs and Hardy and Littlewood found it very frustrating to try to get him to explain any of the results which poured forth.
The Man Who Knew Infinity by Robert Kanigel provides a good insight into the tragic life of Ramanjuan. Other than his remarkable achievement however, Ramanjuan has no great part in the tale of the Riemann Hypothesis, although tables of primes up to ten million amongst his recently uncovered papers, have led to speculation that he may in fact have had a more accurate formula for calculating primes than the one he gave to Hardy.
And Siegel also discovered a formula for calculating the zeros amongst Riemann's papers - it contained the same formula that Hardy had used, but with a refinement. Even 65 years after his death Riemann was still ahead of the game.

For completeness' sake - here is his formula for calculating the zeros:

$$\xi\left(s\right)=\frac{1}{2}s\left(s-1\right)\pi^{-s/2}\Gamma\left(s/2\right)\zeta\left(s\right)$$

$$\xi\left(\frac{1}{2}+it\right)=\int_{u=-\infty}^{\infty}\left(\sum_{n=1}^{\infty}\left(4\pi^2n^4e^{9u/2}-6n^2\pi e^{5u/2}\right)e^{-n^2\pi e^{2u}}\right)e^{iut}du$$

And there have been other proofs - using similar methods to Hardy and Littlewood, Selberg proved that 5-10 per cent of the zeros lie on the critical line, and using the same method Brian Conrey holds the current record with a proof that 40 per cent of zeros lie on the critical line.
Hilbert proposed that a machine could exist that could check whether zeros were on the line. And while his idea was purely imaginary, and before the coming of the code-breakers at Bletchley park, Alan Turing came along not too much later and invented such a machine. The machine would not prove the hypothesis, of course, but it would be far more efficient at finding a zero off the line if there was one. Turing's machine calculated the location of the first 1104 zeros... and then broke.
And then supercomputers such as Cray's came on the scene and hundreds of millions of zeros were calculated. Over ten trillion zeros have been found, and there has been no zero off the line yet to disprove the Hypothesis.

Certainly, the notoriety attainable by proving the hypothesis, and the ensuing royalties etc, and the million dollar prize... these are all incentives, but a proof would have more significant consequences for the world than just these... With the new computer age, as well as the finding of more zeros, the Riemann Hypothesis has become important to more than just number theorists. RSA coding, the basis of internet commerce, relies on the difficulty of factorising large numbers into their prime factors. And while a proof of the Riemann Hypothesis would not lead directly to a method for cracking prime numbers, it is entirely possible that the methods developed to prove the hypothesis would also lead to a far greater understanding of the nature of the primes, and therefore possibly to superior code-breaking methods. Generally though, any reference made to the Hypothesis in relation to internet commerce seems to be just hype by the media.

One of the most unusual, but perhaps the most promising of attempts at proving the Hypothesis, came about almost by pure chance over dinner after a conference. Montgomery, a number theorist who was investigating the gaps between the primes, was introduced to the physicist Dyson by his companion at the table. The small talk eventually turned to what Montgomery was working on and when he explained it to Dyson, Dyson is reported to have been amazed to hear that the patterns that Montgomery had described replicated exactly the behaviour of the Hermitian matrices that he and other physicists were using to calculate the energy levels of heavy nuclei found in certain elements. Odylzko, using a supercomputer, compared the gaps in the matrices and between the zeros, and found that when one reached very high numbers, the similarity waned a little. But physicist Sir Michael Berry looked at Odylzko's work and was able to inform him that the errors between the matrix and zero gaps were not random errors, but rather errors that were due a chaotic quantum effect.

And so, with this connection between physics and the Hypothesis reaffirmed - was there any benefit that could be gleaned from the result of this serendipitious post-prandial convesation?

There was a dearth of progress on the Hypothesis at this time - people were unwilling to share their work because of the prize that was at stake. Brian Conrey persuaded Fry Computers to sponsor a prize for a method of calculating the moments of the zeta function.

It was known that the first and second moments of the function were 1 and 2, respectively, but the methods used for calculating these did not work for other moments. A meeting was held with both physicists and mathematicians in attendance, and the challenge of finding more moments was posed to them.

The following year another meeting was held, where a young physicist named Keating revealed a method that he claimed would predict the moments of the function. His answers for the third and fourth moments matched those which had been independently calculated by some mathematicians, but whereas the mathematicians' result had no satisfactory formula, Keating's method could be used to calculate more moments.

Keatings went on to look at hydrodynamics, and studied the properties of a ball of liquid spinning in a vacuum. What would happen if, for example, a small force were applied to the ball - would it disintegrate, or reform? He perceived a certain resemblance between the inherent stability of these hydrodynamics and the unwillingness of the zeros to stray from the critical line. Interestingly, when Keating went to consult Riemann's work on the primes, he found calculations relating to the same hydrodynamics in the same set of papers! It is quite likely that Riemann had noticed this same connection over one hundred years before...

It seems that Riemann may have discovered far more than anyone has yet guessed - perphaps Alain Connes' current attempt, in using Weil's and Grothendiecke's new mathematical language to make a connection between non-commutative geometry and the primes, and which has already contributed to advances in string theory, may well turn up in some little notebook of Riemann's somewhere. Who knows?

6 Riemann's Prime Counting Function

'The great transformation awaits, Megatron - Prepare to be reformatted.' - Optimus Prime.

Here we shall try to introduce some more of the mathematics that Riemann used in his 1859 paper.
Riemann approached the prime counting function analytically, so if we are to think in the realm of the real numbers, just note that $\pi(8.28893) = \pi(8) = 4$.

$$^{4}J\left(x\right) = \pi\left(x\right) + \frac{1}{2}\pi\left(\sqrt{x}\right) + \frac{1}{3}\pi\left(\sqrt[3]{x}\right) + \frac{1}{4}\pi\left(\sqrt[4]{x}\right) + \frac{1}{5}\pi\left(\sqrt[5]{x}\right) + ...$$

This is not an infinite series - once the roots drop below 2 the value of π will become zero.
Take $x = 1000$ for example:

$$J\left(10000\right) = \pi\left(10000\right) + \frac{1}{2}\pi\left(100\right) + \frac{1}{3}\pi\left(21.54..\right) + \frac{1}{4}\pi\left(10\right) + \frac{1}{5}\pi\left(6.31..\right) + \frac{1}{6}\pi\ 4.64..) + \frac{1}{7}\pi\left(3.73..\right)$$

$$+\frac{1}{8}\pi\left(3.16..\right) + \frac{1}{9}\pi\left(2.78..\right) + \frac{1}{10}\pi\left(2.51..\right) + \frac{1}{11}\pi\left(2.31..\right) + \frac{1}{12}\pi\left(2.15..\right) + \frac{1}{13}\pi\left(2.03..\right) + 0 + 0 + ...$$

which, when you count up the primes gives:

$$J\left(10000\right) =$$

$$(1229) + \frac{(25)}{2} + \frac{(8)}{3} + \frac{(4)}{4} + \frac{(3)}{5} + \frac{(2)}{6} + \frac{(2)}{7} + \frac{(2)}{8} + \frac{(1)}{9} + \frac{(1)}{10} + \frac{(1)}{11} + \frac{(1)}{12} + \frac{(1)}{13}$$

So $J\left(10000\right) = 1247.09799$

Recall when we wrote the sin function and the ζ function as products of their factors. What happens if we were to multiply out the factors of some similar function? Well, some coefficients would be negative and some positive, and some would cancel out with others.
For our example, we want the inverse of the zeta function which returns it's values as dictated by the Mobius function $\mu\left(n\right)$ defined as:

[4] Riemann defined this function as f in his paper, but to save confusion we will follow Harold Edwards' example and refer to it as the 'J' function.

$\mu(n) = 0$ if n has a square factor.
$\mu(n) = -1$ if n is a prime, or the product of an odd number of different primes.
$\mu(n) = 1$ if n is the product of an even number of different primes.

Which gives us another way to write the inverse of the zeta function:

$$\frac{1}{\zeta(s)} = \sum_n \frac{\mu(n)}{n^s}$$

When Mobius inversion is applied to our J function we get:

$$\pi(x) = \sum_n \frac{\mu(n)}{n} J\left(\sqrt[n]{x}\right)$$

So we have $\pi(x)$ in terms of $J(x)$

If we return to our previous example, and wish to calculate the number of primes up to 10000 we can work it out thus:

$$\pi(x) = J(10000) - \frac{1}{2}J(100) - \frac{1}{3}J(21.54..) - \frac{1}{5}J(6.31..) + \frac{1}{6}J(4.64..) - \frac{1}{7}J(3.73..)$$
$$-\frac{1}{10}J(2.51..) - \frac{1}{11}J(2.31..) - \frac{1}{13}J(2.03..) - 0...$$

$$\pi(x) = 1247.09799 - 14.266667 - 3.4880556 - 0.8512 + 0.5128333$$
$$-0.285714286 - 0.2 - 0.1818182 - 0.1538461 = 1229$$

And 1229 *is*, in fact, the number of primes below 10000. Now, of course, since the J function is calculated using the π function - this probably does not seem instantly remarkable.
However - Riemann found a way to express the J function in terms of the ζ function, so here is the connection we were looking for - we should be able to express the prime number function in terms of the zeta function.

Now, how did Riemann do this? Recall

$$\zeta(s) = \prod \left(1 - \frac{1}{p^s}\right)^{-1}$$

Now, taking the log of both sides we get

$$\log \zeta(s) = \log \prod \left(1 - \frac{1}{p^s}\right)^{-1}$$

And on the right-hand side, notice that since the $\log(a \times b = \log a + \log b$ we end up with

$$\log \zeta(s) = \sum_p \log \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\log \zeta(s) = \sum_p - \log \left(1 - \frac{1}{p^s}\right)$$

Recall Newton's definition that

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + ...$$

Leaving us with an infinite sum of infinite sums... Lovely! Consider any one small part, with p for the primes and n for the integers:

$$\frac{1}{n} \times \frac{1}{p^{ns}}$$

Now, take the function x^{-s-1}

$$\int_{p^n}^{\infty} x^{-s-1} dx = \frac{-1}{s} \times \left[\frac{1}{x^s}\right]_{p^n}^{\infty} = \frac{1}{s} \times \frac{1}{p^{ns}}$$

So we can write

$$\frac{1}{n} \times \frac{1}{p^{ns}} = \frac{s}{n} \times \int_{p^n}^{\infty} x^{-s-1} dx$$

Returning to the J-function, $x = p^n$ is where the $J(x)$ takes a step up of $1/n$.
The J-function is made up of these strips.
For example, a strip of height 1 going from each prime off to infinity, a strip of height 1/2 going from the square of each prime off to infinity... a strip of height 1/n going from p^n off to infinity...
So the expression $\int_{p^n}^{\infty} J(x)\, dx$ represents the integral of all the infinite strips at p^n with height $\frac{1}{n}$, which is of course, the entire J-function.
Unfortunately, the area under the J-function is infinite.

Riemann chose to deal with this problem by multiplying $J(x)$ by x^{-s-1}
Since it is no longer infinite, what now is the area under this curve?
What is the value of

$$\int_0^\infty J(x)\, x^{-s-1} dx?$$

Well, now we end up with a decreasing concave function. For every prime there is an infinite sum of

$$\int_{p^n}^\infty \frac{1}{n} \times x^{-s-1} dx$$

$$= \frac{1}{n} \int_{p^n}^\infty x^{-s-1} dx$$

Recall

$$\frac{1}{n} \times \frac{1}{p^{ns}} = \frac{s}{n} \times \int_{p^n}^\infty x^{-s-1} dx$$

So

$$\frac{1}{n} \times \frac{1}{p^{ns}} = s \times \int_0^\infty J(x)\, x^{-s-1} dx$$

$$\int_0^\infty J(x)\, x^{-s-1} dx = \frac{1}{s} \times \frac{1}{n} \times \frac{1}{p^{ns}} = \frac{1}{s} \log \zeta(s)$$

So there we have our ζ function in terms of the J function.

The final result that eventually emerges from this is $J(x)$, from which we can calculate the number of primes up to a given number, expressed in terms of the zeta-function.

$$J(x) = Li(x) - \sum_p Li(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t}$$

where the ρ are the non-trivial zeros of the zeta-function! This shows where the zeros of the zeta-function come into play with regard to counting the primes. *This* is how Riemann used the zeros to correct the error in his estimation of $\pi(n)$.

7 Conclusions

'The cosmical importance of this conclusion is profound and the possibilities it opens for the future very remarkable, greater in fact than any suggested before by science in the whole history of the human race. ' - Francis William Aston.

LaTeXis well worth learning, and not so difficult as it appears at first glance (unlike the Hypothesis - which, while very interesting, is every bit as difficult as one might imagine it to be!)
The paper has shown that Number Theory seems to be inextricably linked with the rest of mathematics, as opposed to being purely an abstract indulgence.
There is an awful lot of information relating to the Riemann Hypothesis, and mathematics in general on the internet. Mind you, there is a lot of awful information as well - any number of 'proofs' of the Riemann Hypothesis can be found.
Riemann's Hypothesis was cited in Hilbert's famous lecture as one of the top problems that faced mathematicians at the turn of the 19th century. The Hypothesis is the only one of those problems that survived to become one of the Millenium Problems with a million-dollar bounty.
And while Riemann's Hypothesis is unlikely to be proven anytime soon, the journey towards the proof will undoubtedly continue to furnish mathematics with a wealth of new interesting ideas and results.
And while anyone can explain the Goldbach Conjecture, hopefully this paper will have provided a succinct and understandable notion of what the Riemann Hypothesis is, and shown why it merits the notoriety associated with it.
The proof that 'all non-trivial zeros of the zeta function have real part one-half' is left to the reader.

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