Computational Homology of Cubical and Permutahedral Complexes

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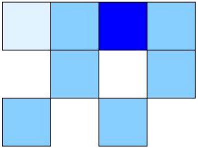
Overview

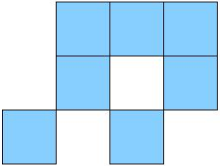
- Illustration of applied topology
- Homology group construction
- The permutahedron
- Contraction
- Implementation
- Discrete vector fields
- Geometric function
- Persistent homology

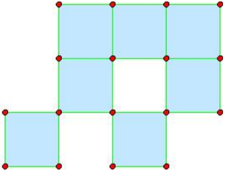
```
gap>A:=ReadImageAsPurePermutahedralComplex("washers.jpg",250);
Pure Permutahedral Complex of dimension 2.

gap> Bettinumbers(A);
[ 14 , 10 , 0 ]
```



$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Two cells are said to be *neighbours* if they share a vertex. The *neighbourhood* of a cell is the union of all its neighbours.

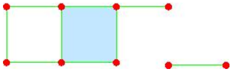




Definition

A *chain complex* is a sequence of free abelian groups $C_0, C_1, C_2, \ldots$ connected by boundary homomorphisms $\delta_n : C_n \to C_{n-1}$ such that the composition of $\delta_n \circ \delta_{n+1} = 0$.

$$\ldots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0$$



Columns in δ_1' correspond to edges and the rows to vertices.

$$\delta_1' = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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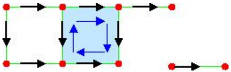
$$\ldots \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} 0$$



$$\delta_1 = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Homology Group Construction

Definition

From its chain complex, we can find the nth *homology group* of X.

$$H_n(X) := \ker(\delta_n)/\mathrm{im}(\delta_{n+1}).$$

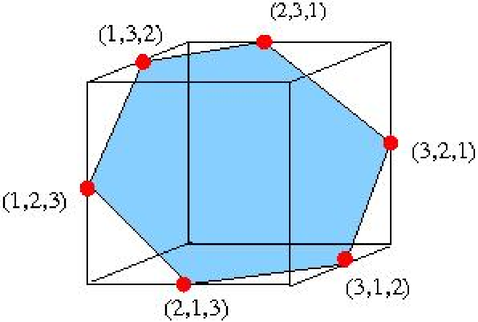
Definition

The n^{th} Betti number is the rank of $H_n(X)$, and gives the number of holes with an n-dimensional boundary in X.

These matrices can be large, (δ_1 is a 22320×15572 matrix for our `washers.jpg` example) so reducing the size of the space without changing its homology aids computation.

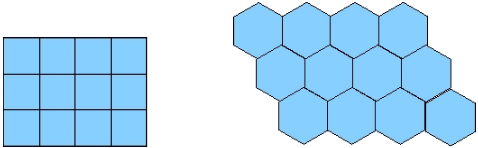
The Permutahedron

An n-dimensional permutahedron is the convex hull of $n!$ points in $(n+1)$-dimensional Euclidean space. The coordinates of these points are the permutations of the integers $(1, 2, 3, \dots, n+1)$.



Two vertices, v_1 and v_2 are connected by an edge if swapping two entries in v_1 whose values differ by 1, gives v_2.

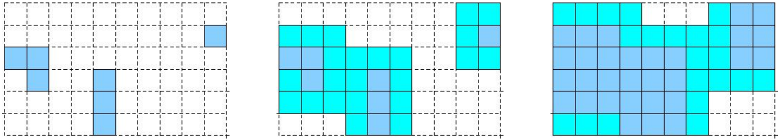
Main Advantage of Permutahedral Tessellation



An n-dimensional permutahedron has $2^{n+1} - 2$ neighbours, whereas an n-cube has $3^n - 1$ neighbours.

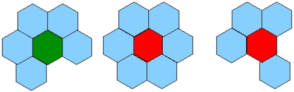
Table: Number of neighbours of an n-cell in dimension n.

n	Cubical	Permutahedral
1	2	2
2	8	6
3	26	14
4	80	30
5	242	62



```
gap> A:=0;;
gap> for i in [1..3] do A:=List([1..100],i->Copy(A));;
gap> C:=PureCubicalComplex(A);;
gap> P:=PurePermutahedralComplex(A);;
gap> for i in [1..20] do
> C:=ThickenedPureCubicalComplex(C); od; time;
92
gap> for i in [1..20] do
> P:=ThickenedPurePermutahedralComplex(P); od; time;
9
```

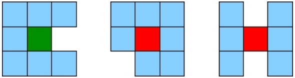
Contraction



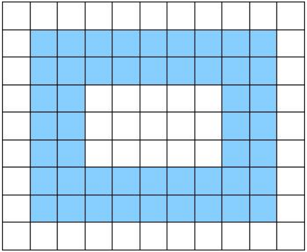
Definition

A cell is said to be *positive* if we can remove it without affecting the homology of its neighbourhood. Removing all positive cells from a space reduces its size without changing its homology.

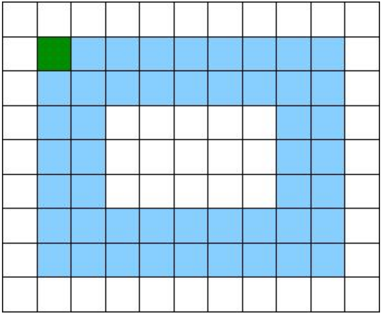
A list indicating whether any possible neighbourhood is positive, in the permutahedral case, is stored in `HAP_Permutahedral` for up to 4 dimensions, allowing for efficient recognition and removal of any positive cells. In the cubical case, this list is only available up to dimension 3, as the list for 4 dimensions would be of length 2^{80}.

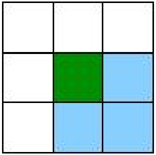


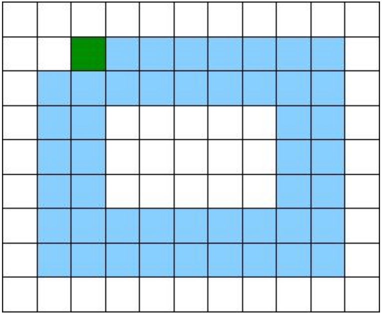
A list $\mathbb{L}^d$ is stored in HAP_Permutahedral with each entry being 1 or 0. The position of each entry corresponds uniquely to a particular neighbourhood. The software need only then check the neighbourhood of a cell in some complex and check whether the corresponding entry in $\mathbb{L}^d$ is 1 or 0, rather than having to calculate the homology of the neighbourhood of each cell. Consider the space below which has 48 facets.

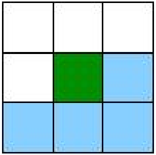


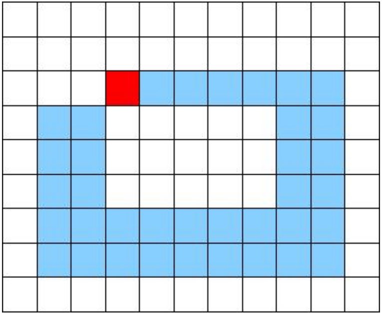
We progress through each facet, checking for facets with positive neighbourhoods. If we find one we remove it and iterate through the space again.

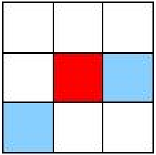




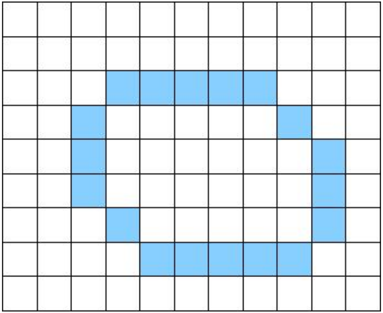






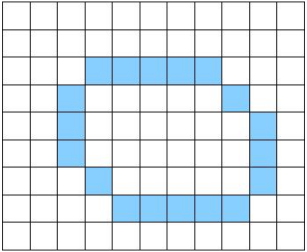


This space X has the same Betti numbers as our original space, but only 18 facets.

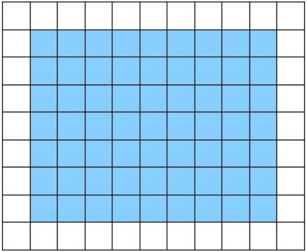


X

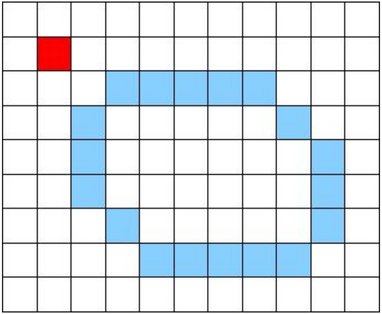
By finding a space B such that $X \subset B$ and $Y \subset B$ such that Y is maximal and homotopy equivalent to X, it is often possible to reduce the size of the necessary space even further.

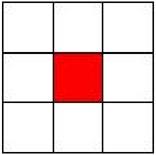


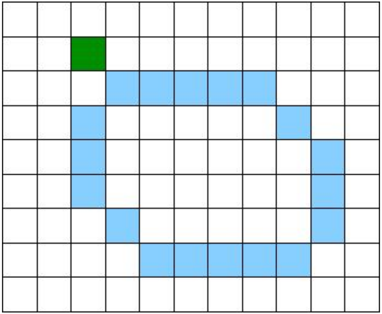
$Y' := X$

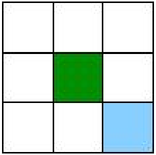


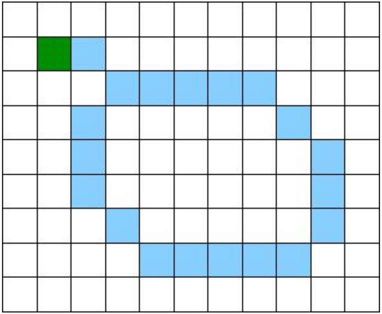
B

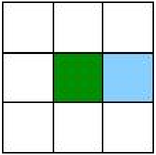


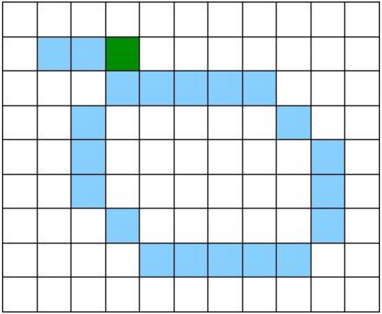


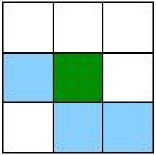


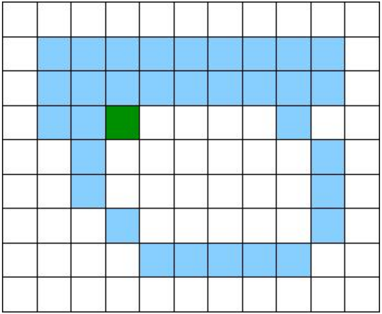


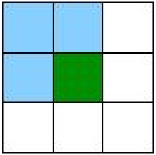


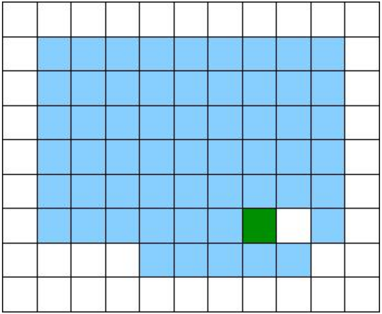


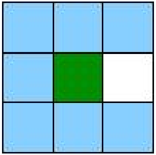


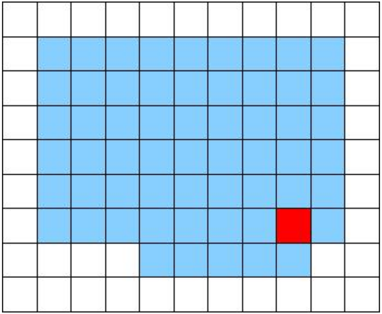


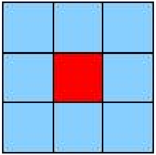


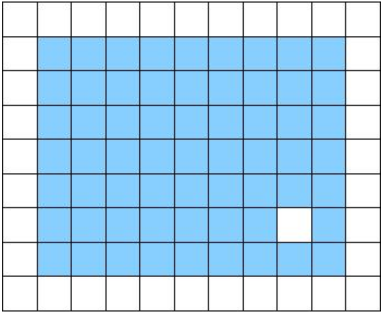






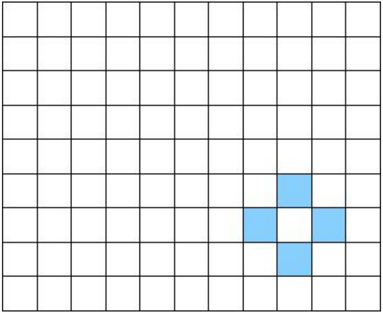




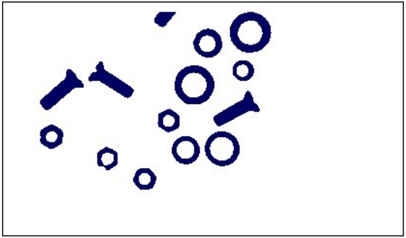


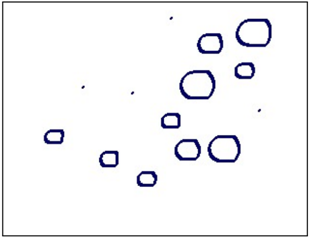
Y

Contracting Y using the first method shown reduces to the space below, with only 4 facets:



```
gap> SizeOfPurePermutahedralComplex(A);
6752
gap> C:=ContractedComplex(A);
Pure Permutahedral Complex of dimension 2.
gap> SizeOfPurePermutahedralComplex(C);
745
gap> ZZ:=ZigZagContractedComplex(A);;
gap> SizeOfPurePermutahedralComplex(ZZ);
64
```





Implementation

Definition

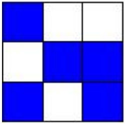
A *pure B-complex* $K = (A, B, \underline{T})$ consists of

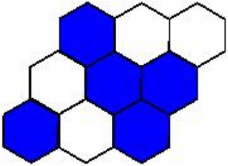
- a binary array $A = (a_\lambda)$ of dimension d,
- a basis $\underline{T} = \{t_1, t_2, \ldots, t_d\}$ for $\mathbb{R}^d$,
- a finite set $B \subset \mathbb{Z}^d$. We call B the *ball*.

Definition

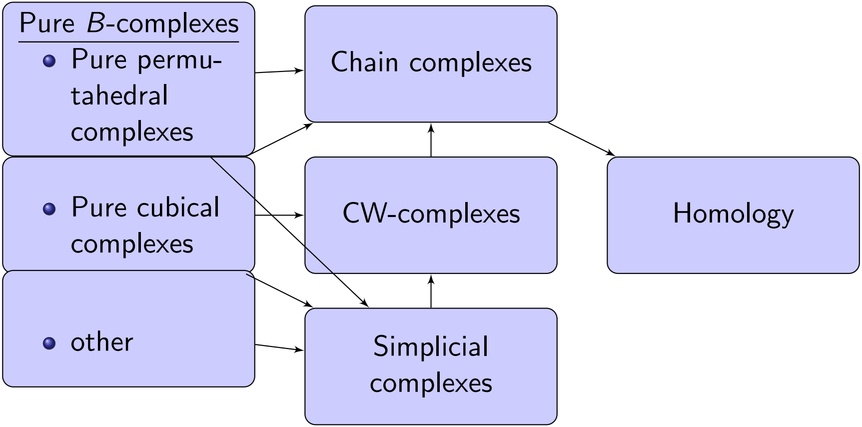
Given some $\lambda \in \mathbb{Z}^d$ such that $k_\lambda = 1$, where k_λ is a cell in K, we define the *neighbourhood* $N(k_\lambda)$ to be the union of $k_{\lambda+b} : b \in B$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$





Data Types

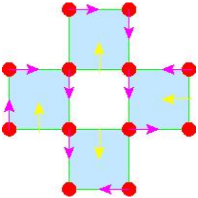


Pure B-complexes
Pure permutahedral complexes
Pure cubical complexes
other
Chain complexes
CW-complexes
Simplicial complexes
Homology

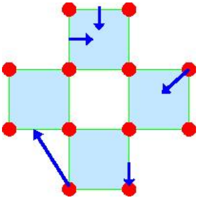
Definition

A *discrete vector field* on a regular CW-space X is a collection of arrows $\alpha : s \to t$ where

- s, t are cells and any cell is involved in at most one arrow
- $dim(t) = dim(s+1)$
- s lies in the boundary of t.



Discrete vector field

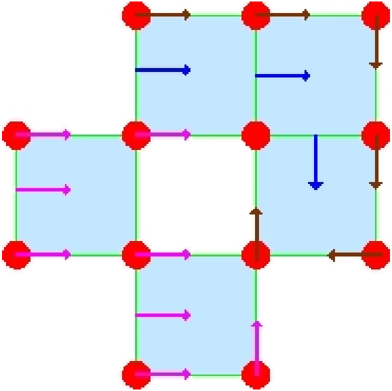


Not a discrete vector field

Definition

A sequence of arrows $\alpha_1 : s_1 \to t_1, \alpha_2 : s_2 \to t_2, \ldots$ in a vector field is a *path* if

- the s_i are cells of common dimension d and the t_i are cells of common dimension $d + 1$.
- each s_{i+1} lies in the boundary of t_i.



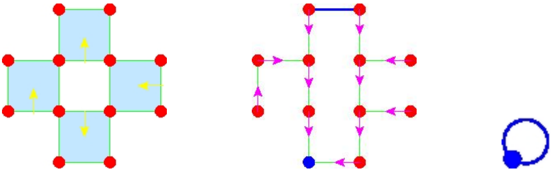
A discrete vector field is *admissible* if it contains no cyclic paths. A cell is *critical* if it is involved in no arrows of the vector field. A fundamental theorem of discrete Morse theory states:

Theorem

If X is a regular CW-space with an admissible discrete vector field then there is a homotopy equivalence

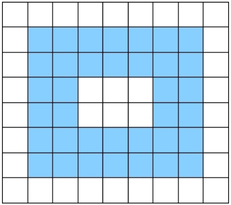
$$X \simeq Y$$

where Y is a CW-space whose cells are in one-to-one correspondance with the critical cells of X.

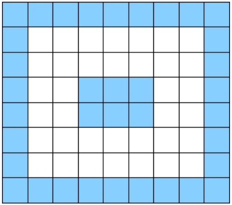


Another benefit of using permutahedra

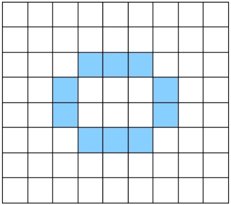
Another benefit to using permutahedral complexes is that, unlike in the cubical case, $Comp(Cont(x)) \simeq Cont(Comp(x))$.



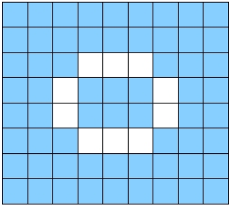
X



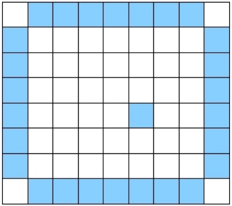
$\mathrm{Comp}(X)$



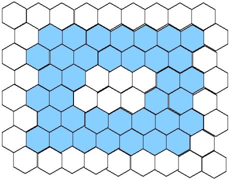
$\mathrm{Cont}(X)$



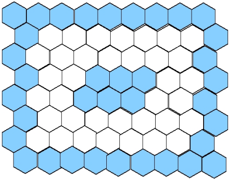
$\mathrm{Comp}(\mathrm{Cont}(X))$



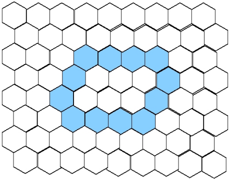
$\mathrm{Cont}(\mathrm{Comp}(X))$



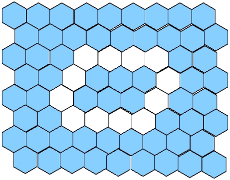
X



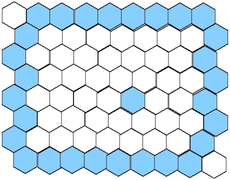
$\mathrm{Comp}(X)$



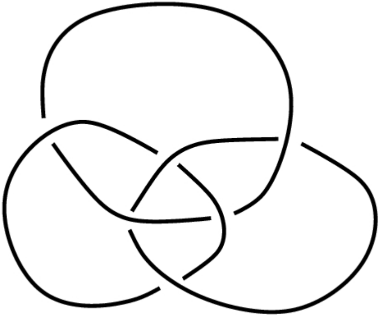
$\mathrm{Cont}(X)$



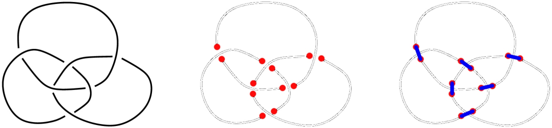
$\mathrm{Comp}(\mathrm{Cont}(X))$



$\mathrm{Cont}(\mathrm{Comp}(X))$



B : A link diagram of Borromean rings.



Cubical	Size	Time	Permutahedral	Size	Time
			Contracted(B)	3238	33
Crop(B)	33654	4	Crop		1
Complement	1025724	3	Complement	352642	1
Contracted	114556	259	Contracted	70716	30
ToCWComplex		63	ToCWComplex		32
CriticalCells		33	CriticalCells		13

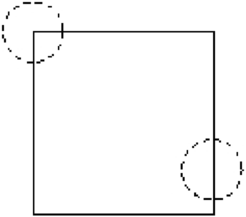
```
gap> F:=FundamentalGroupOfRegularCWComplex;
there are 3 generators and 2 relators of total length 22
<fp group of size infinity on the generators [ f1, f2,
f3 ]>
gap> RelatorsOfFpGroup(F);
```

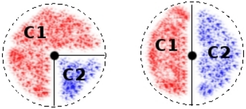
$[f1 * f3^-1 * f2^-1 * f3 * f2 * f1^-1 * f2^-1 * f3^-1 * f2 * f3,$
$f1^-1 * f3^-1 * f2 * f1 * f3 * f1^-1 * f3^-1 * f2^-1 * f3^2 * f1 * f3^-1]$

If we let p_r be a ball of radius r about a point p on the boundary $B(X)$ of some space X, and let C_1 and C_2 be the two components of p_r as divided by $B(X)$, then if

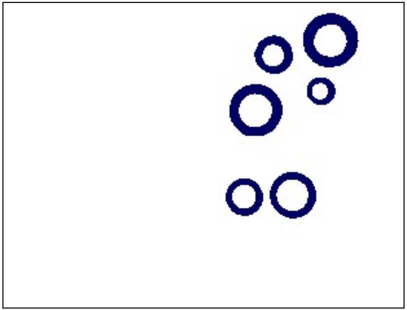
$$\frac{\text{Size}(C_1) - \text{Size}(C_2)}{\text{Size}(C_1) + \text{Size}(C_2)} < \tau$$

where τ is some user-defined threshold, we can say the point p is a singularity. This can be used to find 'interesting' points in a space.





```
gap> numberofwashers:=0;
gap> for i in [1..PathComponentOfPureCubicalComplex
(A,0)] do
> if Size(SingularitiesOfPureCubicalComplex
(PathComponentOfPureCubicalComplex(A,i),5,50))=0 then
> numberofwashers:=numberofwashers+1; fi; od;
gap> numberofwashers;
6
```



Estimating Homology from a Data Sample

There are two types of people in the world; those who can extrapolate from incomplete data...

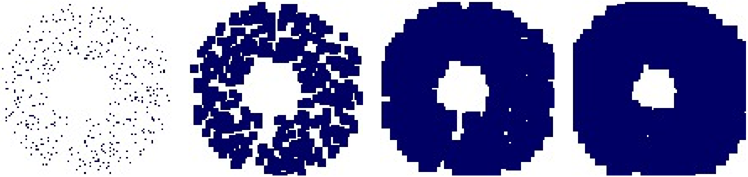
```
gap> S:=Sample(A,400);;
gap> Bettinumbers(S);
[317 , 0 , 0 ]
```

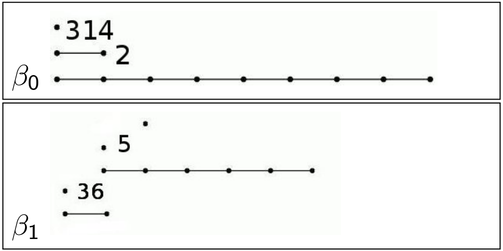
```
gap> T:=Thickened(S);;
gap> Bettinumbers(T);
[3 , 37 , 0 ]
```

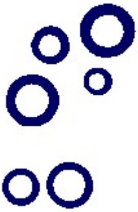
Definition

A *bar code* is a graph which has horizontal edges and vertices arranged in columns. If a vertex in the ith column is connected to a vertex in the jth column, this signifies that the hole *persists* from the ith to the jth thickening. A hole which persists through a significant number of thickenings can be deemed representative of the space.

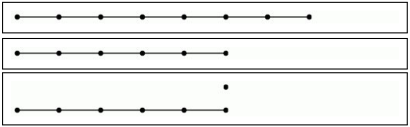
Persistent Homology

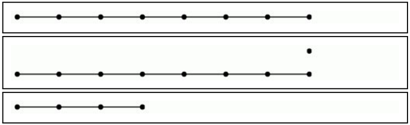






β_1 for each path component





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